

Overlap, disorder, and directed polymers: A renormalization group approach

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The overlap of a $(d+1)$ -dimensional directed polymer of length t in a random medium is studied using a renormalization group approach. In $d > 2$ it vanishes at T_c for $t \rightarrow \infty$ as $t^{-\Sigma}$ where $\Sigma = [(d-1)/(3-2d)](d/z)$ and z is the transverse spatial rescaling exponent. The same formula holds in $d=1$ for any finite temperature and it agrees with previous numerical simulations at $d=1$. This value of Σ for $d > 2$ is up to one loop but an exact scaling relation is obtained. We also obtain the scaling exponent for mutual repulsion of two chains in the random medium and the behavior of overlap around T_c .

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One commonly studied quantity in random systems is the overlap of appropriate physical variables in different states as, for example, the overlap of magnetizations in two states for a spin glass system [1,2]. The distribution function for the overlap q , $p(q)$, which, e.g. in a pure Ising system has two δ function peaks for the two possible low temperature ferromagnetic states, has a nontrivial structure in the spin glass phase because of the broken ergodicity. Such nontriviality of the overlap distribution is the main characteristic for spin glass type systems. Notwithstanding the importance of overlaps, very little is understood analytically about it.

Of late, a directed polymer (DP) in a random environment is taken to be the paradigm of disordered systems. This is because of the strong analogy in behavior with other random systems and spin glass [3], and more so because of the availability of exact results through the Bethe ansatz [4], use of various analytical and numerical methods like nonlinear differential equation [5], the transfer matrix [6–8], and other approaches. Even in this situation, the question of overlap remains a mind boggling issue [7,8]. In this paper, we implement a renormalization group (RG) approach to obtain the scaling behavior of the overlap. To our knowledge, no such analytical result for overlap is known for any other realistic random system.

The $(d+1)$ -dimensional directed polymers are random walks stretched along a particular direction with fluctuations in the transverse d -dimensional space. These stringlike objects appear in various random systems of interface fluctuations and pinning [4], crystal growth [5], spin glasses [9], etc. Most of the recent work done on this problem attempts to understand the low temperature (strong disorder) phase which, as a matter of fact, is the only possible phase in $1+1$ dimensions. The problem in $1+1$ dimensions is almost settled with relevant exponents known exactly [10]. The two important exponents are χ and $\zeta=1/z$, which describe the free energy fluctuation and transverse size as the length $t \rightarrow \infty$, namely $f \sim t^{\chi/z}$ and $\langle x \rangle \sim t^{\zeta}$. For $d=1$, $\chi=1/2$ and $z=3/2$, with $\chi+z=2$. However, precise values of these exponents in higher dimensions and their exact dimensional

dependences are not yet well understood mainly because of the lack of any perturbative fixed point [11]. For high enough dimensions ($d > 2$) it is found that there is a phase transition [12,13] from the high temperature (weak disorder) to the low temperature phase (strong disorder). What happens at $d=2$ is not clear [8,7], though the consensus seems to be against a phase transition [5,7]. We mention, in passing, that several exact results are known, especially in connection with such disorder induced phase transitions, if the randomness is in the interaction instead of the medium [14].

The high temperature phase of a directed polymer in a random potential in $d > 2$ is simple since there the quenched and annealed free energies are equal. The situation is more complex in the low temperature phase because of the non-zero overlap and the subsequent nonanalyticity of the free energy that supports the hypothesis of the coexistence of several pure states [9,3]—a phenomena reminiscent of spin glasses. Partial information regarding the thermal, geometrical properties and a few related distribution functions in the low temperature phase are available from Monte Carlo simulations, expansion methods and transfer matrix techniques [13,8]. However, at present the overlap, especially near the critical point, seems to elude these techniques. In this paper our main focus is at this transition temperature. Unlike the tree problem [15] which, in some way, corresponds to the mean field limit $d \rightarrow \infty$, our result is true for finite dimensions. [See, e.g., Ref. [16] for the peculiarities of the Cayley tree problem.]

Similar to the concept of the overlap of two different magnetization states [1], here in the DP picture, the overlap means the number of common spatial points visited by two different configurations of the polymer. Introducing two different configurations of a polymer is equivalent to starting with the original polymer with a replica. The overlap is then the average number of contacts of these two polymers (see below for a more precise definition). The procedure for the evaluation of the overlap would be to introduce a new interaction that penalizes such contacts with coupling constant v_0 . The overlap follows, as statistical mechanics prescribes, from the calculation of the appropriate derivative of the free energy of such an interacting system.

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In the path integral representation the working Hamiltonian for two interacting directed polymers

$$\mathcal{H} = \int_0^t d\tau \left[\sum_{i=1,2} \left(\frac{\gamma}{2} \dot{\mathbf{x}}_i^2(\tau) + \frac{\lambda}{2\gamma} V(\mathbf{x}_i(\tau), \tau) \right) + \frac{\lambda}{2\gamma} v_0 \delta(\mathbf{x}_{12}(\tau)) \right] \quad (1)$$

where $\mathbf{x}_i(t)$ is the d -dimensional spatial coordinate of the i th chain at the contour length t , $\dot{\mathbf{x}}_i(t) = d\mathbf{x}_i(t)/dt$, and \mathbf{x}_{12} is the relative separation of the two chains. The first two terms represent the entropic fluctuations of two free Gaussian chains with γ as the bare line tension. V corresponds to the space and time dependent random potential experienced by the two chains interacting at the same t with δ function potential of strength $\lambda v_0/(2\gamma)$. The significance of the peculiar factor $\lambda/2\gamma$ with the random potential and with the coupling constant v_0 will be clear from discussions later. The random potential is taken to be uncorrelated, normally distributed [5] with

$$\overline{V(\mathbf{x}, \tau)V(\mathbf{y}, \tau')} = 2\Delta \delta(\mathbf{x} - \mathbf{y}) \delta(\tau - \tau'), \quad (2)$$

where the overbar stands for the averaging over the disorder. The disorder is quenched so that the average of $\ln W$, where W is the appropriate partition function, is needed.

The overlap for the above system in the continuum limit can be precisely defined as

$$q(t) = \frac{1}{t} \int_0^t d\tau \delta(\mathbf{x}_{12}(\tau)). \quad (3)$$

It can be obtained from the relation

$$q(t) = -\frac{1}{t} \left. \frac{df_2(v_0, t)}{dv_0} \right|_{v_0=0},$$

where $f_2(v_0, t)$ is the free energy for the Hamiltonian in Eq. (1). A scaling form

$$f_2(v_0, t) = t^{\chi/z} f(v_0 t^{-\phi/z}) \quad (4)$$

is expected with ϕ determining the crossover exponent. This implies $q = t^{\Sigma} Q(v_0 t^{-\phi/z})$ where

$$\Sigma = (\chi - \phi - z)/z. \quad (5)$$

This particular problem in a discrete version at $d=1$ has been studied by Mezard numerically in [3]. His simulation results are consistent with $\zeta = 1/z = 2/3$, $\chi = 1/2$, and $\phi/z = -2/3$. A corollary is that the behavior of one chain remains unaffected by the presence of the other. This need not be surprising because the ‘‘screening’’ type effects in ordinary self and mutually avoiding polymers are finite density phenomena. Since at $d=1$, $\chi - \phi - z = 0$, one obtains $q(v_0, t) \sim q(v_0 t^{-\phi/z})$. One of our aims is to determine ϕ .

Here we use the continuum formulation and map the problem to a KPZ type nonlinear differential equation for the free energy [5,17]. A dynamic renormalization group approach, in the manner of Ref. [17], following a perturbative

calculation in a Fourier conjugate space, is developed to study this equation. In the process, we obtain the scaling exponent for the interaction and establish that in any arbitrary dimensions χ and z remain the same as those of the single chain problem, as one would expect, even in the presence of the interaction. A series for the renormalized coupling constant can be identified by collecting the appropriate terms from the perturbative series. The recursion relation for the coupling constant, found after the use of the momentum shell technique, manifests the scaling form of the mutual interaction and the overlap. The exponents obtained through this process are for the critical point T_c for $d > 2$ but for the finite temperature phase for $d=1$. An appeal to finite size scaling then enables us to extend the result to $T \neq T_c$ for $d > 2$.

We consider two chains which are tied at one end ($t=0$) at the origin of the d -dimensional space and extended up to \mathbf{x}_1 and \mathbf{x}_2 at length t . The partition function $W(\mathbf{x}_1, \mathbf{x}_2, t)$, which is basically a sum of the Boltzmann weights of all configurations of two such chains, can be written in the path integral form as

$$W(\mathbf{x}_1, \mathbf{x}_2, t) = \int_{(0,0,0)}^{(\mathbf{x}_1, \mathbf{x}_2, t)} \mathcal{D}\mathbf{x}'_1 \mathcal{D}\mathbf{x}'_2 \exp[-\mathcal{H}], \quad (6)$$

where $\int \mathcal{D}\mathbf{x}'_1 \mathcal{D}\mathbf{x}'_2$ stands for all possible paths of the two polymers and \mathcal{H} is given by Eq. (1). This implies that the partition function satisfies a Schrödinger type equation written suppressing the argument of W

$$\frac{\partial}{\partial t} W = \left[\gamma \sum_{i=1,2} \nabla_i^2 + \frac{\lambda}{2\gamma} g_0(\mathbf{x}_1, \mathbf{x}_2, t) \right] W, \quad (7)$$

where $g_0(\mathbf{x}_1, \mathbf{x}_2, t) = V(\mathbf{x}_1, t) + V(\mathbf{x}_2, t) + v_0 \delta(\mathbf{x}_{12})$ appear as the potential.

Our approach starts with another version of the above equation for the free energy $h(\mathbf{x}_1, \mathbf{x}_2, t) = (2\gamma/\lambda) \ln W(\mathbf{x}_1, \mathbf{x}_2, t)$ which satisfies, again suppressing the arguments,

$$\frac{\partial}{\partial t} h = \sum_{i=1,2} \left[\gamma \nabla_i^2 h + \frac{\lambda}{2} (\nabla_i h)^2 \right] + g_0. \quad (8)$$

The impressive feature of this equation is that the parameter λ , which was previously controlling the random potential and the mutual interaction between the chains in the original Hamiltonian in Eq. (1), now appears only with the nonlinear term. Equation (8) can be decoupled when there is no mutual interaction between the chains and such a decoupled equation can be solved exactly when $\lambda=0$ [18]. One can then make a small perturbation in the nonlinearity (i.e., in λ) which is effectively equivalent to introducing a small disorder into the problem. The recursion relation for v_0 , obtained after perturbation in the nonlinearity λ , gives the influence of disorder on v_0 .

A glance at Eq. (8) shows that under the transformation $x \rightarrow bx$ and $t \rightarrow b^2 t$ the parameters of the equation change as

$$(\gamma, \lambda, v_0) \rightarrow (b^{z-2} \gamma, b^{\chi+z-2} \lambda, b^{z-d-\chi} v_0). \quad (9)$$

Therefore in the absence of the nonlinearity, i.e., when $\lambda=0$, $z=z_0=2$ and $\chi=2-d$ keep γ and v_0 invariant. This scaling furthermore ensures the speciality of $d=d_c=2$ since for $d<d_c$ a small amount of nonlinearity becomes relevant with the growth of the length scale. Such speciality of d_c is reflected later in the perturbative series. At this level one finds the necessity of going beyond the simple scaling analysis to carry out the RG analysis since the scaling dimension of v_0 , $z-\chi-d$, vanishes at $d=1$, while numerically it is found to be -1 [3].

The formal solution of Eq. (8) in $(\mathbf{K}, \mathbf{k}, \omega)$ space, Fourier conjugate to $(\mathbf{x}_1, \mathbf{x}_2, t)$, is given by

$$\begin{aligned} h(\mathbf{K}, \mathbf{k}, \omega) &= G_0(\mathbf{K}, \mathbf{k}, \omega) g_0(\mathbf{K}, \mathbf{k}, \omega) - (\lambda/2) G_0(\mathbf{K}, \mathbf{k}, \omega) \\ &\times \int_{p, q, \Omega} [\mathbf{p} \cdot (\mathbf{K} - \mathbf{p}) + \mathbf{q} \cdot (\mathbf{k} - \mathbf{q})] h(\mathbf{p}, \mathbf{q}, \Omega) \\ &\times h(\mathbf{K} - \mathbf{p}, \mathbf{k} - \mathbf{q}, \omega - \Omega), \end{aligned} \quad (10)$$

where $G_0(\mathbf{K}, \mathbf{k}, \omega) = [\gamma(\mathbf{K}^2 + \mathbf{k}^2) - i\omega]^{-1}$ represents the bare propagator and

$$\int_{p, q, \Omega} = \int \frac{d\Omega}{2\pi} \frac{d\mathbf{p} d\mathbf{q}}{(2\pi)^{2d}}.$$

The fact that the random potential and the interaction are in the same footing in the above equation is now utilized in defining the effective propagator $G(\mathbf{K}, \mathbf{k}, \omega)$ and the effective coupling constant v as

$$\begin{aligned} h(\mathbf{K}, \mathbf{k}, \omega) &= G(\mathbf{K}, \mathbf{k}, \omega) [V(\mathbf{K}, \omega) \delta(\mathbf{k}) + V(\mathbf{k}, \omega) \delta(\mathbf{K}) \\ &+ v \delta(\mathbf{K} + \mathbf{k}) \delta(\omega)]. \end{aligned} \quad (11)$$

Note that such a restriction on momenta is imposed automatically by the right-hand side (RHS) of Eq. (10).

Now we are in a position to initiate the perturbative series, the terms of which after disorder averaging lead to "closed loop diagrams." [See, e.g., Ref. [17].] Here we shall consider terms up to $O(\lambda^2)$ and $O(v_0)$. Collecting the appropriate terms from the perturbative series for the renormalized propagator satisfying either $\mathbf{K}=0$ or $\mathbf{k}=0$, one obtains a series identical to the renormalized propagator for a single chain in a random medium [5,17]. It becomes evident from the series that there is no contribution at $O(\lambda)$ since such terms either do not have any loop or even if they do, they contribute to $O(v_0^2)$, a higher order term which we are not considering here. For convenience, we cite the series obtained for a single chain propagator with momentum variable \mathbf{K}

$$\begin{aligned} G(\mathbf{K}, \mathbf{k}, \omega) &= G_0(\mathbf{K}, \mathbf{k}, \omega) + C(-\lambda/2)^2 G_0^2(\mathbf{K}, \mathbf{k}, \omega) (2\Delta) \\ &\times \int_{q, \Omega} \mathbf{q} \cdot (\mathbf{K} - \mathbf{q}) \mathbf{q} \cdot \mathbf{K} G_0(\mathbf{K} - \mathbf{q}, \mathbf{k}, \omega - \Omega) \\ &\times G_0(\mathbf{q}, 0, \Omega) G_0(-\mathbf{q}, 0, -\Omega) \end{aligned} \quad (12)$$

where $C=4$ counts all the possible ways of noise contraction and $\mathbf{k}=0$ in this case. The terms which contribute to the renormalization of the vertex follow the constraint $\mathbf{K} + \mathbf{k} = 0$. One can write down the series for $vG(\mathbf{K}, -\mathbf{K}, 0)$

to obtain an equation in which the vertex and the propagator $G(\mathbf{K}, -\mathbf{K}, 0)$ renormalization take place in a combined fashion and interestingly can also be isolated. To $O(\lambda^2)$ the series for the renormalized propagator $G(\mathbf{K}, -\mathbf{K}, 0)$ is the same as Eq. (12) with $C=8$ and $\mathbf{k}=-\mathbf{K}$ and v is given by the series

$$\begin{aligned} v &= v_0 + 8 \left(-\frac{\lambda}{2} \right)^2 (2v_0 \Delta) \int_{q, \Omega} [\mathbf{q} \cdot (\mathbf{K} - \mathbf{q})]^2 G_0(\mathbf{q}, -\mathbf{K}, \Omega) \\ &\times G_0(\mathbf{K} - \mathbf{q}, 0, -\Omega) \times G_0(\mathbf{q}, -\mathbf{q}, 0) G_0(0, \mathbf{q} - \mathbf{K}, \Omega). \end{aligned} \quad (13)$$

In a diagrammatic representation the second term on the RHS of this equation would correspond to an exchange type diagram. At face value, the series of $G(\mathbf{K}, -\mathbf{K}, 0)$ does not resemble the series for the single chain propagator $G(\mathbf{K}, 0, \omega)$, but it is easy to show that there is no change in the renormalization of the line tension γ from that of Ref. [5] in the long wavelength limit. This, furthermore, confirms that in any arbitrary dimensions, so far as the free energy, lateral extension are concerned, exponents do not change even after the inclusion of the mutual interaction.

The second term of Eq. (13) involves a momentum integration with an upper cutoff Λ which indicates the resolution up to which the system is probed. To study the variation of v with length scale, we execute an RG procedure consisting of two steps: (i) an integration over the momentum shell between the momenta Λ and $\Lambda \exp(-\delta l)$ and (ii) the momentum rescaling $\mathbf{k} \rightarrow \mathbf{k} \exp(-\delta l)$ which restores the upper cutoff Λ as before. Now after carrying out the first step and the integration over Ω we obtain an effective coupling constant (in the long wavelength limit) differing from v_0 by a term $K_d \bar{\lambda}^2 v_0 \delta l / 2$. Here $K_d = S_d / (2\pi)^d$ with S_d as the surface area of the unit d -dimensional sphere and $\lambda^2 = (\lambda^2 \Delta) / \gamma^3$ is dimensionless. This additional term is the fluctuation contribution of the disorder and crucial for the RG analysis. A little manipulation after the rescaling [step (ii)] produces the recursion relation

$$\frac{dv}{dl} = (z - \chi - d)v + \frac{\bar{\lambda}^2 v K_d}{2}, \quad (14)$$

with $v=v_0$ when $l=0$. The recursion for $\bar{\lambda}$ is quoted below from Ref. [5],

$$\frac{d\bar{\lambda}}{dl} = \frac{2-d}{2} \bar{\lambda} + K_d \frac{(2d-3)}{4d} \bar{\lambda}^3, \quad (15)$$

which has a fixed point $\bar{\lambda}^* = \{2d(2-d)/[(3-2d)K_d]\}^{1/2}$. At $d=1$ it is a stable fixed point which shows that the phase is influenced by the disorder at all temperatures. There is no physical fixed point for $1.5 < d < 2$. The nontrivial fixed point becomes unstable for $d > 2$. From the flow, one concludes that this unstable fixed point corresponds to a critical point that separates the two phases dominated by the entropy (high temperature phase) or by the disorder (low temperature phase). The situation at $d=2$ is more complicated since disorder is marginally relevant [17,19]. Using the relation $\chi + z = 2$, which is a consequence of Galilean invariance, and $\lambda = \bar{\lambda}^*$ we find from Eq. (14)

$$\phi = (2 + d - 2z) + d(d - 2)/(3 - 2d). \quad (16)$$

At $d=1$ exact values $z=3/2$ and $\chi=1/2$ yield the scaled variable as $\nu t^{2/3}$. The remarkable feature is that in Eq. (14) the term $z - \chi - d$, which originates from the simple scaling analysis of Eq. (9), vanishes at $d=1$ and the entire t dependence of the scaled variable comes only from the fluctuation part. The scaling exponent of ν matches the numerical prediction of Mezard at $d=1$ [3]. Now, using the exponent ϕ , Eq. (16), we find

$$\Sigma = d(d - 1)/[z(3 - 2d)]. \quad (17)$$

Since our interest is in the $\nu_0=0$ situation, it is possible to write down the *exact* exponent relation from Eqs. (14) and (5) as

$$\Sigma = [(\bar{\lambda}^* K_d)/2 - d]/z \quad (18)$$

in terms of the single chain fixed point, which is remarkable.

Going back to the original problem of a single DP, the overlap is obtained by setting $\nu_0=0$. We therefore obtain

$$q \sim t^\Sigma, \quad (19)$$

where, as mentioned before, this is the overlap in the low temperature phase for $d < 1.5$ but at the critical point for $d > 2$. The exponent Σ vanishes for $d=1$, as has been found in Ref. [3]. Since $\Sigma < 0$ for $d > 2$, the overlap disappears at the transition temperature when the thermodynamic limit is approached—as one should expect.

We extend our $T=T_c$ result to the critical region by invoking a finite size scaling hypothesis [20], $q = \xi^{\Sigma_1} Q(t/\xi)$ where ξ is the longitudinal (parallel to t) correlation length and $\xi \sim |T - T_c|^{-\nu}$ near the critical point. Comparing with Eq. (19), we find $\Sigma_1 = \Sigma$, so that, in the thermodynamic limit

$$q \sim |T - T_c|^{-\nu\Sigma} \quad (20)$$

as $T \rightarrow T_c$. Unfortunately the value of ν is still not known with confidence [7,8].

It is possible to explain the effect of random environment more physically. In the situation where the disorder dominates the physics, the chain is swollen to take advantage of the occasional traps that lower the energy. The loss in the entropy is offset by the gain in the energy, yielding $\zeta > 1/2$. In this scenario, it is therefore expected that the repulsion with another chain will have no significant effect. Hence to the leading order, ν_0 is not to have any effect on the renormalization of the properties of the polymers. However, with two chains there will be a certain amount of overlap in their attempts to take advantage of the same traps. On a bigger scale, such closely spaced traps would appear as an interaction between the chains. This leads to the renormalization of ν_0 changing its scaling exponent.

In summary, we have shown through one-loop RG analysis that the behavior of a single chain in the random medium remains unaffected even if we introduce another chain interacting with it through a short-range repulsive interaction. At $d=1$ our results show a finite overlap at all temperatures indicating a strong coupling phase and the exponents match exactly Mezard's conjecture based on numerical simulations. In higher dimensions ($d > 2$), we evaluated the scaling exponent of the overlap at $T=T_c$. By using a finite size scaling ansatz, we thereafter predict the temperature dependence of the overlap as $T \rightarrow T_c$.

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[1] M. Mezard *et al.*, *Spinglass Theory and Beyond* (World Scientific, Singapore, 1987).
 [2] G. Parisi, *Phys. Rev. Lett.* **50**, 1946 (1983).
 [3] M. Mezard, *J. Phys. (Paris)* **51**, 1831 (1990).
 [4] M. Kardar, *Nucl. Phys.* **B290**, 582 (1987).
 [5] M. Kardar, G. Parisi, and Y. C. Zhang, *Phys. Rev. Lett.* **56**, 889 (1986).
 [6] M. Kardar, *Phys. Rev. Lett.* **55**, 2923 (1985).
 [7] J. M. Kim, A. J. Bray, and M. A. Moore, *Phys. Rev. A* **44**, R4782 (1991).
 [8] B. Derrida and O. Golinelli, *Phys. Rev. A* **41**, 4160 (1990).
 [9] B. Derrida and H. Sphon, *J. Stat. Phys.* **51**, 817 (1988).
 [10] D. A. Huse, C. L. Henley, and D. S. Fisher, *Phys. Rev. Lett.* **55**, 2924 (1985).
 [11] T. Ala-Nissila, T. Hjelt, J. M. Kosterlitz, and O. Venäläinen, *J. Stat. Phys.* **72**, 207 (1993) and references therein.

[12] J. Z. Imbrie and T. Spencer, *J. Stat. Phys.* **52**, 609 (1988).
 [13] J. Cook and B. Derrida, *J. Stat. Phys.* **57**, 89 (1989).
 [14] S. M. Bhattacharjee and S. Mukherji, *Phys. Rev. Lett.* **70**, 49 (1993); S. Mukherji and S. M. Bhattacharjee, *Phys. Rev. E* **48**, 3483 (1993).
 [15] J. Cook and B. Derrida, *Europhys. Lett.* **10**, 195 (1989).
 [16] D. S. Fisher and D. A. Huse, *Phys. Rev. B* **43**, 10 728 (1991).
 [17] E. Medina, T. Hwa, M. Kardar, and Y. C. Zhang, *Phys. Rev. A* **39**, 3053 (1989).
 [18] S. F. Edwards and D. R. Wilkinson, *Proc. R. Soc. London Ser. A* **381**, 17 (1982).
 [19] Lei-Han Tang, T. Natterman, and B. M. Forrest, *Phys. Rev. Lett.* **65**, 2422 (1990).
 [20] See, e.g., M. N. Barber, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 1983), Vol. 7.